Exercise 14

Solve the initial-value problem.

$$9y'' + y = 3x + e^{-x}, \quad y(0) = 1, \quad y'(0) = 2$$

Solution

This is a linear inhomogeneous ODE, so its general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$9y_c'' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \to \quad y_c' = re^{rx} \quad \to \quad y_c'' = r^2 e^{rx}$$

Substitute these formulas into equation (1).

$$9(r^2 e^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$9r^2 + 1 = 0$$

Solve for r.

$$r^2 = -\frac{1}{9}$$

$$r = \left\{-\frac{i}{3}, \frac{i}{3}\right\}$$

Two solutions to the ODE are $e^{-ix/3}$ and $e^{ix/3}$. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned} y_c &= C_1 e^{-ix/3} + C_2 e^{ix/3} \\ &= C_1 \left(\cos\frac{x}{3} - i\sin\frac{x}{3}\right) + C_2\left(\cos\frac{x}{3} + i\sin\frac{x}{3}\right) \\ &= (C_1 + C_2)\cos\frac{x}{3} + (-iC_1 + iC_2)\sin\frac{x}{3} \\ &= C_3\cos\frac{x}{3} + C_4\sin\frac{x}{3} \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$9y_p'' + y_p = 3x + e^{-x} \tag{2}$$

Since the inhomogeneous term is the sum of a polynomial of degree 1 and an exponential, the trial function is $y_p = (Ax + B) + Ce^{-x}$.

$$y_p = Ax + B + Ce^{-x} \quad \to \quad y_p' = A - Ce^{-x} \quad \to \quad y_p'' = Ce^{-x}$$

Substitute these formulas into equation (2).

$$9(Ce^{-x}) + (Ax + B + Ce^{-x}) = 3x + e^{-x}$$

Simplify the left side.

$$Ax + B + 10Ce^{-x} = 3x + e^{-x}$$

Match the coefficients on both sides to get a system of equations for A, B, and C.

$$\begin{aligned} A &= 3 \\ B &= 0 \\ 10C &= 1 \end{aligned}$$

Solving it yields

$$A = 3 \quad \text{and} \quad B = 0 \quad \text{and} \quad C = \frac{1}{10}.$$

The particular solution is then

$$\begin{aligned} y_p &= (Ax + B) + Ce^{-x} \\ &= 3x + \frac{1}{10}e^{-x}. \end{aligned}$$

As a result, the general solution to the original ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_3 \cos\frac{x}{3} + C_4 \sin\frac{x}{3} + 3x + \frac{1}{10}e^{-x}. \end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = -\frac{C_3}{3}\sin\frac{x}{3} + \frac{C_4}{3}\cos\frac{x}{3} + 3 - \frac{1}{10}e^{-x}$$

Apply the initial conditions to determine C_3 and C_4.

$$\begin{aligned} y(0) &= C_3 + \frac{1}{10} = 1 \\ y'(0) &= \frac{C_4}{3} + 3 - \frac{1}{10} = 2 \end{aligned}$$

Solving this system yields

$$C_3 = \frac{9}{10} \quad \text{and} \quad C_4 = -\frac{27}{10}.$$

Therefore,

$$y(x) = \frac{9}{10}\cos\frac{x}{3} - \frac{27}{10}\sin\frac{x}{3} + 3x + \frac{1}{10}e^{-x}.$$

Below is a plot of the solution versus x.